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On Brane-World Cosmology

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Abstract

In the study of three-brane cosmological models, an unusual law of cosmological expansion on the brane has been reported. According to this law, the energy density of matter on the brane quadratically enters the right-hand side of the new equations for the brane world, in contrast with the standard cosmology, where it enters the similar equations linearly. However, this result is obtained in the absence of curvature-dependent terms in the action for the brane. In this paper, we derive the field equations for a brane world embedded into a five-dimensional spacetime in the case where such terms are present. We also discuss some cosmological solutions of the resulting equations.

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I. INTRODUCTION

The string-theory inspired idea of a four-dimensional universe as a brane world embedded into a multi-dimensional universe has now become very popular. In many papers, models of such kind are considered in the four-dimensional cosmological context; in particular, the theory of cosmological perturbations within the frames of such models is currently being developed (for a comprehensive list of references see, e.g., [1]). In several papers [2–5], an unusual law of cosmological expansion of a four-dimensional universe embedded into (or bounding [5]) a five-dimensional space has been reported. According to this law, the energy density of matter on the brane enters quadratically the right-hand side of the new equations for the brane world, in contrast with the standard cosmology, where it enters the similar equations linearly. Such a behaviour might strongly modify the standard cosmological model [4]. However, this result is obtained in the absence of curvature-dependent terms in the action for the four-dimensional brane world. The purpose of this article is to derive the field equations for a brane world embedded into (or bounding) a five-dimensional space in the case where such curvature-dependent terms are present. After that, we consider a particular example of a cosmological situation and discuss some of its solutions.

II. FIELD EQUATIONS

Consider a theory with a four-dimensional hypersurface (brane) Σ which is the boundary of a five-dimensional manifold $\mathcal{M}$.[1] We take the action of the theory to have the natural form

$$S = M_5^3 \left[\int_{\mathcal{M}} \left({}^{(5)}R - 2\Lambda_5 \right) + 2 \int_{\Sigma} K \right] + \int_{\mathcal{M}} L_5(g_{ab}, \Phi) + M_4^2 \int_{\Sigma} \left({}^{(4)}R - 2\Lambda_4 \right) + \int_{\Sigma} L_4(h_{ab}, \phi) \, . \tag{1}$$

Here, ${}^{(5)}R$ is the scalar curvature of the Lorentzian five-dimensional metric g_{ab} on $\mathcal{M}$, and ${}^{(4)}R$ is the scalar curvature of the induced metric $h_{ab} = g_{ab} - n_a n_b$ on Σ, where n^a is the vector field of the outer unit normal to Σ. The boundary Σ is assumed to be timelike, so that the vector field n^a is spacelike. The quantity $K = K_{ab}h^{ab}$ is the trace of the symmetric tensor of extrinsic curvature $K_{ab} = h^c{}_a \nabla_c n_b$ of Σ in $\mathcal{M}$. The quantities $L_5(g_{ab}, \Phi)$ and $L_4(h_{ab}, \phi)$ denote, respectively, the Lagrangian densities of the five-dimensional matter fields Φ and of the four-dimensional matter fields ϕ whose dynamics is restricted to the boundary Σ so that they interact only with the induced metric h_{ab}. Note that some of the fields ϕ in principle may represent restrictions of some of the fields Φ to the boundary Σ. All integrations over $\mathcal{M}$ and over Σ are taken, respectively, with the natural volume elements $\sqrt{-g}\, d^5x$ and $\sqrt{-h}\, d^4x$, where g and h are, respectively, the determinants of the matrices of components

[1] In fact, all the equations of this section are valid for a n-dimensional space $\mathcal{M}$ bounded by a $(n-1)$-dimensional hypersurface Σ.

of the metric on $\mathcal{M}$ and of the induced metric on Σ in a coordinate basis. The constants M_n and Λ_n denote, respectively, the n-dimensional Planck mass and cosmological constant.

In this paper, we freely use the notation and conventions of [6]. In particular, following Sec. 10.2 of [6], we use the one-to-one correspondence between tensors in Σ and tensors in $\mathcal{M}$ that are invariant under projection to the tangent space to Σ, i.e., tensors $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ such that

$$T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = h^{a_1}{}_{c_1}\cdots h^{a_k}{}_{c_k} h_{b_1}{}^{d_1}\cdots h_{b_l}{}^{d_l} T^{c_1\cdots c_k}{}_{d_1\cdots d_l} \, . \tag{2}$$

The third term in (1) containing the four-dimensional curvature is often missing from the action, or its contribution is missing from the equations of motion. However, in general, this term seems to be essential since it is generated as a quantum correction to the matter action in (1). Note that this quantum correction typically involves an infinite number of terms of higher order in curvature (a similar situation in the context of the AdS/CFT correspondence is described in [7]). Thus, we assume that such terms are present and retain only the lowest-order ones in (1).

In this paper, we are interested only in the metric equation, so the Lagrangians for the matter fields Φ and ϕ will not be specified. The first variation of action (1) with respect to the metric g_{ab} is equal to[2]

$$\begin{aligned} \delta S = M_5^3 \int_{\mathcal{M}} \left({}^{(5)}G_{ab} + \Lambda_5 g_{ab} \right) \delta g^{ab} - \int_{\mathcal{M}} T_{ab} \delta g^{ab} + M_5^3 \int_{\Sigma} S_{ab} \delta h^{ab} \\ + M_4^2 \int_{\Sigma} \left({}^{(4)}G_{ab} + \Lambda_4 h_{ab} \right) \delta h^{ab} - \int_{\Sigma} \tau_{ab} \delta h^{ab} \, , \end{aligned} \tag{3}$$

where ${}^{(n)}G_{ab}$ denotes the n-dimensional Einstein's tensor, $S_{ab} \equiv K_{ab} - K h_{ab}$, and T_{ab} and τ_{ab} define, respectively, the five-dimensional and four-dimensional stress-energy tensors of matter. Note that the variation δh^{ab} on the brane is not an independent quantity, but is completely and uniquely determined by δg^{ab}. For simplicity, we also assume that the Lagrangian $L_5(g_{ab}, \Phi)$ does not contain derivatives of the metric g_{ab}, the presence of which might contribute to the surface terms in (3).

On an extremal field configuration, variation (3) is equal to zero for arbitrary variations of the metric g_{ab}. Considering variations that leave the induced metric on Σ intact, i.e., for which $h^a{}_c h^b{}_d \delta h^{cd} \equiv 0$ and, hence, the surface integrals in (3) vanish, we obtain the equation of motion in the five-dimensional bulk:

$${}^{(5)}G_{ab} + \Lambda_5 g_{ab} = \frac{1}{M_5^3} T_{ab} \, . \tag{4}$$

It is important to stress that the Gibbons–Hawking boundary term [the second term in the square brackets in action (1)] is required to obtain this equation in a consistent way [8].

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[2]Those interested in the derivation of (3) may look into the appendix.

Now, considering arbitrary variations of the metric g_{ab} and taking into account equation (4), we obtain the equation of motion on the boundary Σ in the form

$$ {}^{(4)}G_{ab} + \Lambda_4 h_{ab} + M S_{ab} = \frac{1}{M_4^2} \tau_{ab} \,, \tag{5} $$

where $M = M_5^3 / M_4^2$. It is the presence of the tensor S_{ab} in the equation of motion (5) that makes the dynamics on the brane Σ unusual.

One of the Gauss–Codacci relations, namely,

$$ D_a S^a{}_b = {}^{(5)}R_{cd} n^d h^c{}_b \,, \tag{6} $$

where D_a is the (unique) derivative on the brane Σ associated with the induced metric h_{ab}, together with equation (5) and the bulk equation (4) imply the relation

$$ D_a \tau^a{}_b = T_{cd} n^d h^c{}_b \,. \tag{7} $$

Thus, the four-dimensional stress-energy tensor is covariantly conserved if and only if the right-hand side of (7) is vanishing at the brane, in particular, if the stress-energy tensor T_{ab} of the five-dimensional matter is proportional to g_{ab} or to h_{ab} at the brane.

Thus far, we considered Σ to be a boundary of a five-dimensional manifold $\mathcal{M}$. However, the theory can easily be extended to the case where Σ is embedded into $\mathcal{M}$. In this case, it can be regarded as a common boundary of two pieces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$, and we should simply add the actions of the form of the first term in (1) for these two pieces. In varying the resulting action, we must respect the condition that the metrics induced on the brane Σ by the metrics of these two pieces coincide; however, the extrinsic curvatures of Σ in $\mathcal{M}_1$ and in $\mathcal{M}_2$ are allowed to be different. Equation (4) remains valid in the bulk, and equation (5) will be modified to

$$ {}^{(4)}G_{ab} + \Lambda_4 h_{ab} + M \left(S^{(1)}_{ab} + S^{(2)}_{ab} \right) = \frac{1}{M_4^2} \tau_{ab} \,, \tag{8} $$

where the tensors $S^{(1)}_{ab}$ and $S^{(2)}_{ab}$ are constructed with the use of the respective extrinsic curvatures. The analog of equation (7) also can easily be derived.

If several branes are embedded into the manifold $\mathcal{M}$, equations of the form (8) are valid for each of these branes. If a brane is a common boundary of more than two bulk manifolds, the corresponding number of similar terms will be present inside the brackets of (8).

Neglecting the third term in action (1) amounts to taking the limit of $M_4 \to 0$. In this case, equation (8) reduces to the Israel's junction condition [9]

$$ M_5^3 \left(S^{(1)}_{ab} + S^{(2)}_{ab} \right) = \tau_{ab} \,. \tag{9} $$

On the other hand, taking the limit of $M_5 \to 0$ is equivalent to setting $M = 0$ in equation (5) or (8). In this limiting case, the influence of the five-dimensional bulk vanishes, and we obtain the standard equation of general relativity.

III. COSMOLOGICAL EXAMPLE

As an illustration, we consider a particular example of the cosmological situation described in [5]. We take the five-dimensional metric in the static spherically-symmetric form

$$ds_5^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega_{(3)} \, , \tag{10}$$

where $d\Omega_{(3)}$ is the metric of the unit three-sphere. In the case of vanishing stress-energy tensor of the five-dimensional matter, as a solution of (4) we can take

$$f(r) = 1 - \alpha r^2 \, , \tag{11}$$

where $\alpha = \Lambda_5/6$. The brane Σ is taken to be spherically-symmetrically embedded into this manifold according to the law $r = a(t)$. We then discard the exterior $r > a(t)$ and consider the resulting space with Σ as a boundary. The tensor S_{ab} for this boundary can easily be calculated. Its nonzero components have the form (see also [5])

$$S^0{}_0 = -\frac{3\sqrt{f(a) + \dot{a}^2}}{a} \, , \qquad S^i{}_j = -\,\delta^i{}_j \frac{1}{a^2 \dot{a}} \frac{d}{d\tau} \left(a^2 \sqrt{f(a) + \dot{a}^2} \right) , \tag{12}$$

where $i, j = 1, 2, 3$ label the coordinates on the unit three-sphere and the overdot denotes the derivative with respect to the cosmological time τ on the brane, in terms of which the induced metric is given by the line element

$$ds_4^2 = -\left[f(a) - \frac{1}{f(a)} \left(\frac{da}{dt} \right)^2 \right] dt^2 + a^2 d\Omega_{(3)} = -d\tau^2 + a^2 d\Omega_{(3)} \, . \tag{13}$$

Equation (5) then yields the following two equations:

$$\frac{1 + \dot{a}^2}{a^2} + \frac{M\sqrt{f(a) + \dot{a}^2}}{a} = \lambda + \kappa\rho \, , \tag{14}$$

$$\frac{2\ddot{a}}{a} + \frac{1 + \dot{a}^2}{a^2} + \frac{M}{a^2 \dot{a}} \frac{d}{d\tau} \left(a^2 \sqrt{f(a) + \dot{a}^2} \right) = 3\lambda - 3\kappa p \, , \tag{15}$$

where we made the notation $\lambda = \Lambda_4/3$, $\kappa = 1/3M_4^2$, and ρ and p denote the standard components of the four-dimensional stress-energy tensor (that may include its own cosmological-constant contribution). Introducing the Hubble parameter $H \equiv \dot{a}/a$, we have from (14)

$$H^2 + \frac{f(a)}{a^2} = \left[\left(\frac{M^2}{4} + \frac{f(a) - 1}{a^2} + \lambda + \kappa\rho \right)^{1/2} - \frac{M}{2} \right]^2 . \tag{16}$$

Substituting this into equation (15), we obtain the conservation law

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{17}$$

in accordance with the general equation (7). Together with the equation of state (e.g., in the form $p = w\rho$), equations (16), (17) constitute a closed system of cosmological equations.

If we neglect the third term in action (1) and thus take the limit of $M_4 \to 0$, equation (16) will become

$$H^2 + \frac{f(a)}{a^2} = \left(\frac{\rho}{3M_5^3} \right)^2 , \tag{18}$$

with the quadratic dependence of the right-hand side on the energy density, noted previously [2–5]. This result is approximately valid for finite values of M_4 provided $\left| (f(a) - 1)/a^2 + \lambda \right| \ll \kappa\rho \ll M^2$.

In the limit of $M_5 \to 0$, which corresponds to $M \to 0$, the influence of the five-dimensional bulk vanishes, and we recover the equation of the standard cosmology

$$H^2 + \frac{1}{a^2} = \lambda + \kappa\rho\,. \tag{19}$$

This equation is approximately valid for finite values of M_5 provided $M^2 \ll (f(a) - 1)/a^2 + \lambda + \kappa\rho$.

We see that, in principle, the standard regime (19) might be realised at the early stages of the universe expansion, the standard theory of nucleosynthesis thus remaining intact, while, at later stages, the universe might evolve according to the nonstandard law (18).

The system of equations (14), (15) with the function $f(r)$ given by (11) admits a solution of the static empty (that is, $\rho = p = 0$) universe with the scale factor

$$a_0 = \left(\alpha + \frac{M^2}{4} \right)^{-1/2} , \tag{20}$$

provided the value of M_5 (hence, also of M) is negative and the values of parameters are tuned so that

$$\lambda = \alpha - \frac{M^2}{4}\,. \tag{21}$$

Note that, in the case of negative Λ_5, solution (10), (11) describes the anti-de Sitter five-dimensional space. In this case, the value of $\alpha = \Lambda_5/6$ is negative, so that the size of the four-dimensional static universe given by (20) may be arbitrarily large, because the value of $\alpha + M^2/4$ may be arbitrarily close to zero. This is another instance of fine-tuning, similar to that discussed in [10], that countervails the effect of the four-dimensional cosmological constant which, according to (21), must be negative in the present case.

It should be noted, however, that, under relation (21), the system of equations (14), (15) is degenerate at the point

$$a = a_0 \,, \qquad \dot{a} = 0 \,, \tag{22}$$

in the sense that the second-order derivative term in (15) vanishes at this point. This circumstance, in particular, has a consequence that the Cauchy problem (22) for system (14), (15) is ill-defined since, besides the static solution $a \equiv a_0$, it also has the solution

$$a(\tau) = a_0 \cosh\left(\frac{\tau}{a_0}\right) \tag{23}$$

describing the de Sitter spacetime.

Gravitational perturbations of solutions to theory (1) and the resulting effective four-dimensional theory of gravity remain to be investigated. In this respect, some important results in the context of the AdS/CFT correspondence are discussed in [7].

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APPENDIX: VARIATION OF THE ACTION FOR GRAVITY

Here, we derive the expression for the first variation of the action for gravity

$$S_g = \int_{\mathcal{M}} R + 2 \int_{\Sigma} K \,. \tag{A1}$$

Equations of this section will be valid for arbitrary dimension of spacetime. In contrast with the standard derivation, here we do not assume that the variation of g_{ab} vanishes at the boundary Σ, which is taken to be timelike.

We start from the standard expression (see, e.g., Appendix E of [6])

$$\delta\left(\int_{\mathcal{M}} R\right) = \int_{\mathcal{M}} G_{ab}\, \delta g^{ab} + \int_{\mathcal{M}} \nabla^a v_a \,, \tag{A2}$$

where

$$v_a = \nabla^b \left(\delta g_{ab}\right) - g^{cd} \nabla_a \left(\delta g_{cd}\right) \,. \tag{A3}$$

The second integral in (A2) can be transformed with the use of the Stokes theorem as

$$\int_{\mathcal{M}} \nabla^a v_a = \int_{\Sigma} v_a n^a \,, \tag{A4}$$

where

$$v_a n^a = n^a g^{bc} \left[\nabla_c \left(\delta g_{ab} \right) - \nabla_a \left(\delta g_{bc} \right) \right] = n^a h^{bc} \left[\nabla_c \left(\delta g_{ab} \right) - \nabla_a \left(\delta g_{bc} \right) \right] . \tag{A5}$$

Then we have

$$\delta K = \delta \left(h^a{}_b \nabla_a n^b \right) = \delta h^a{}_b \nabla_a n^b + h^a{}_b (\delta C)^b{}_{ac} n^c + h^a{}_b \nabla_a \delta n^b , \tag{A6}$$

where

$$(\delta C)^b{}_{ac} = \frac{1}{2} g^{bd} \left[\nabla_a \left(\delta g_{cd} \right) + \nabla_c \left(\delta g_{ad} \right) - \nabla_d \left(\delta g_{ac} \right) \right] . \tag{A7}$$

The first term in the right-hand side of (A6) is identically zero. Indeed, we have $\delta n_a = - n_a n_b \delta n^b$, so that

$$\begin{aligned} \delta h^a{}_b \nabla_a n^b &= - \left(\delta n^a n_b + n^a \delta n_b \right) \nabla_a n^b = - \left(\delta n^a - n^a n_c \delta n^c \right) n_b \nabla_a n^b \\ &= - \delta n^c h^a{}_c n_b \nabla_a n^b = - \delta n^c n_b K_c{}^b = 0 . \end{aligned} \tag{A8}$$

Thus, variation of the second term of (A1) is

$$\delta \left(2 \int_\Sigma K \right) = \int_\Sigma \left[n^c h^{ab} \nabla_c \left(\delta g_{ab} \right) + 2 h^a{}_b \nabla_a \delta n^b - K h_{ab} \delta h^{ab} \right] , \tag{A9}$$

where the last term in the square brackets stems from the variation of the volume element $\sqrt{-h}\, d^4 x$ in the integral over Σ.

The total boundary term in the variation of action (A1) is given by the sum of (A4) and (A9) with the result

$$(\text{Boundary term}) = \int_\Sigma \left[n^a h^{bc} \nabla_c \left(\delta g_{ab} \right) + 2 h^a{}_b \nabla_a \delta n^b - K h_{ab} \delta h^{ab} \right] . \tag{A10}$$

We transform the first term in the integrand of the last expression:

$$n^a h^{bc} \nabla_c \left(\delta g_{ab} \right) = h^{bc} \nabla_c \left(n^a \delta g_{ab} \right) - h^{bc} \nabla_c n^a \delta g_{ab} = h^{bc} \nabla_c \left(n^a \delta g_{ab} \right) + K_{ab} \delta h^{ab} . \tag{A11}$$

Then

$$(\text{Boundary term}) = \int_\Sigma \left[h^{bc} \nabla_c \left(n^a \delta g_{ab} \right) + 2 h^a{}_b \nabla_a \delta n^b \right] + \int_\Sigma \left(K_{ab} - K h_{ab} \right) \delta h^{ab} . \tag{A12}$$

Now we show that the integrand of the first integral in (A12) is a divergence, so that this integral vanishes for variations of g_{ab} with compact support in Σ. Indeed,

$$\begin{aligned} & h^{bc} \nabla_c \left(n^a \delta g_{ab} \right) + 2 h^a{}_b \nabla_a \delta n^b = h^{bc} \nabla_c \left(\delta n_b - g_{ab} \delta n^a \right) + 2 h^a{}_b \nabla_a \delta n^b \\ & = h^a{}_b \nabla_a \left(g^{bc} \delta n_c + \delta n^b \right) = h^a{}_b \nabla_a \left(h^b{}_c \delta n^c \right) = D_b \left(h^b{}_c \delta n^c \right) , \end{aligned} \tag{A13}$$

where D_a is the (unique) derivative on Σ associated with the induced metric h_{ab}, and the last equality in (A13) is valid by virtue of Lemma 10.2.1 of [6].

As a final result, we have

$$\delta S_g = \int_{\mathcal{M}} G_{ab}\, \delta g^{ab} + \int_\Sigma \left(K_{ab} - K h_{ab} \right) \delta h^{ab} . \tag{A14}$$

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